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NONLINEAR PROBLEM OF HEAT CONDUCTION FOR A HEAT-SENSITIVE SPHERE

Yu. M. Kolyano, E. G. Ivanik, and O. V. Sikora

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The nonlinear problem of heat conduction is solved for a sphere which exchanges heat through its surface with the external medium according to Newton's law. The temperature field is investigated.

Consider a heat-sensitive sphere which has an initial temperature t_{in} and through whose spherical surface r = R heat is exchanged with the external medium, whose temperature jumps from 0 to t_0. The nonstationary temperature field determined by a prescribed heating action is determined by solving the following boundary-value problem:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\lambda(t)\frac{\partial t}{\partial r}\right] = c_v(t)\dot{t}, \tag{1}$$

$$t|_{\tau=0} = t_{in}, \left.\frac{\partial t}{\partial r}\right|_{r=0} = 0, \left.\left\{\lambda(t)\frac{\partial t}{\partial r} + \alpha\left[t - t_0 S_+(\tau)\right]\right\}\right|_{r=R} = 0. \tag{2}$$

Introducing Kirchhoff's substitution

$$\vartheta = \frac{1}{\lambda_0}\int_{t_{in}}^{t}\lambda(\zeta)\,d\zeta \tag{3}$$

and making the assumption that $c_v(t)/\lambda(t) = a \cong$ const, we convert the nonlinear boundary-value problem (1)-(2) into the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\vartheta}{\partial r}\right] = \frac{\dot{\vartheta}}{a}, \tag{4}$$

$$\vartheta|_{\tau=0} = 0, \left.\frac{\partial\vartheta}{\partial r}\right|_{r=0} = 0, \left.\left\{\lambda_0\frac{\partial\vartheta}{\partial r} + \alpha\left[t - t_0 S_+(\tau)\right]\right\}\right|_{r=R} = 0, \tag{5}$$

where λ_0 is the reference value of the thermal conductivity.

The boundary condition at the boundary r = R contains the function t(R, τ), i.e., the solution of the desired nonlinear problem, taken on the boundary surface. It is obvious that t(R, τ) is a function of time only. For this reason, we approximate it with the help of splines of zero order, i.e., with the help of asymmetric unit step functions, as follows:

$$t(R, \tau) = t_{in} + \sum_{i=1}^{m} t_i S_-(\tau - \tau_i), \tag{6}$$

where t_i, where i = 1, ..., m, are, as yet, unknown values, and τ_i are the partition points on the straight line (0, τ).

We note that the representation (6) is adequate for determining the step functions introduced in [1]. Since any continuous and monotonic function is the uniform limit of step functions, the expression (6) is legitimate and correct.

Institute of Applied Problems in Mechanics and Mathematics, Academy of Sciences of the Ukrainian SSR, L'vov. I. Franko Drogobychskii State Pedogogical Institute. Translated from Inzhenerno-fizicheskii Zhurnal, Vol. 62, No. 1, pp. 126-129, January, 1992. Original article submitted March 13, 1991.

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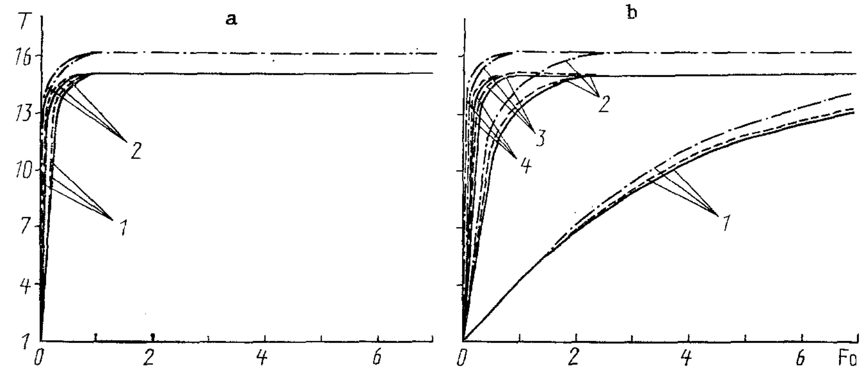

Fig. 1. Dimensionless temperature T as a function of Fourier's number Fo: a) Bi = 10; 1) ρ = 0; 2) 1; b) ρ = 0; 1) Bi = 0.1; 2) 1; 3) 10; 4) 100.

Substituting Eq. (6) into the boundary conditions (5) we arrive at the following boundary-value problem with boundary conditions of the second kind for determining the Kirchhoff variable:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \vartheta}{\partial r}\right)=\frac{\dot{\vartheta}}{a}, \tag{7}$$

$$\vartheta|_{\tau=0}=0,\ \left.\frac{\partial\vartheta}{\partial r}\right|_{r=0}=0,\ \lambda_0\left.\frac{\partial\vartheta}{dr}\right|_{r=R}=-q(\tau), \tag{8}$$

where

$$q(\tau)=\alpha\left[t_{\text{in}}-t_0S_+(\tau)+\sum_{i=1}^{m}t_iS_-(\tau-\tau_i)\right].$$

The solution obtained for the boundary-value problem (7)-(8) using the Laplace transform has the form

$$\vartheta(r,\ \tau)=\text{Bi}\left\{[t_{\text{in}}-t_0S_+(\tau)]\,\Phi\left(\frac{r}{R},\ \text{Fo}\right)+\sum_{i=1}^{m}t_i\Phi\left(\frac{r}{R},\ \text{Fo}-\text{Fo}_i\right)\right\}. \tag{9}$$

Here

$$\Phi\left(\frac{r}{R},\ \text{Fo}\right)=\left[0{,}1\left(3-5\frac{r^2}{R^2}\right)-3\text{Fo}+2\sum_{n=1}^{\infty}\frac{R\sin\mu_n\frac{r}{R}}{r\mu_n^3\cos\mu_n}\exp(-\mu_n^2\text{Fo})\right]S_+(\text{Fo}),$$

$\text{Fo}=\frac{a\tau}{R^2},\ \text{Fo}_i=\frac{a\tau_i}{R^2},\ \text{Bi}=\frac{\alpha R}{\lambda_0}$ and μ_n are the roots of the transcendental equation tan μ = μ.

Many authors [2-4], who linearize the nonlinear boundary-value problem with the help of Kirchhoff's variable, take the boundary condition at the boundary r = R in the following form:

$$\lambda_0\left.\frac{\partial\vartheta}{\partial r}\right|_{r=R}+\alpha\,[\vartheta|_{r=R}-t_0S_+(\tau)]=0, \tag{10}$$

i.e., they approximately set $t \cong \vartheta$, which, of course, is not entirely legitimate.

In this case, the solution of the differential equation (4) with the boundary condition (10) will have the following form:

$$\vartheta_*(r,\ \tau)=t_0\left[1-\sum_{k=1}^{\infty}A_k\frac{R\sin\beta_k\frac{r}{R}}{r\beta_k}\exp(-\beta_k^2\text{Fo})\right], \tag{11}$$

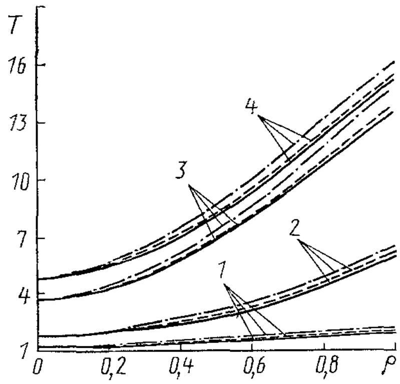

Fig. 2. Dimensionless temperature T as a function of the radial coordinate ρ with Fo = 0.1 and Bi = 0.1 (1), 1 (2), 10 (3), and 100 (4).

where $A_k = 2\frac{\sin\beta_k - \beta_k\cos\beta_k}{\beta_k - \sin\beta_k\cos\beta_k}$ and in addition β_k are determined from the transcendental equation $(1 - \text{Bi})\tan\beta = \beta$.

In order to investigate the temperature field, determined by a prescribed heating action, it is necessary to have a specific temperature dependence of the thermal conductivity. As an example, we take a linear function [4]

$$\lambda(t) = \lambda_0(1 - k_0 t), \tag{12}$$

where k_0 = 0.00006 1/°C, λ_0 = 50.2 W/(m·°C).

Substituting Eq. (12) into Eq. (3) and carrying out the required transformations, we obtain an expression for the desired temperature field in the heat-sensitive sphere, expressed in terms of the Kirchhoff's variable

$$t(r,\ \tau) = \frac{1}{k_0}[1 - \sqrt{(1 - k_0 t_{\text{in}})^2 - 2k_0\vartheta(r,\ \tau)}\ ]. \tag{13}$$

The expression (13) contains the unknown quantities t_i. They can be determined from the following relation:

$$\sum_{i=1}^{m} t_i^* S_-(\text{Fo} - \text{Fo}_i) = \gamma - \frac{1}{K_*}\sqrt{\Psi(1,\ \text{Fo}) - 2K_*\text{Bi}\sum_{i=1}^{m} t_i^*\Phi(1,\ \text{Fo} - \text{Fo}_i)}. \tag{14}$$

Here $\gamma = K_*^{-1} - 1, K_* = k_0 t_{\text{in}}, t_i^* = \frac{t_i}{t_{\text{in}}}$, $\Psi(1,\ \text{Fo}) = (1 - K_*)^2 - 2K_*\text{Bi}\left(1 - \frac{t_0}{t_{\text{in}}}\right)\Phi(1,\ \text{Fo})$.

Setting in Eq. (14) $\text{Fo} = \text{Fo}_i$ $(i = \overline{1,\ m})$ we obtain the following recurrence relations for determining the values of t_i^*:

$$t_i^* = \gamma - K_*^{-1}\sqrt{\Psi(1,\ \text{Fo}_i) - 2K_*\text{Bi}\sum_{j=1}^{i-1} t_j^*\Phi(1,\ \text{Fo}_i - \text{Fo}_j)\,S_+(\text{Fo}_i - \text{Fo}_j)} - \sum_{j=1}^{i-1} t_j^* S_+(\text{Fo}_i - \text{Fo}_j). \tag{15}$$

Thus, substituting the values found for t_i^* into the expression (9) and then substituting Kirchhoff's variable $\vartheta(r,\ \tau)$ into Eq. (13), we determine the desired temperature field in the heat-sensitive sphere. We note that in order to find t_i^* we start from the following considerations: the number of terms in the expansion (6) is determined by the condition

$$|t_{i+1}^* - t_i^*| \leqslant \varepsilon,$$

where $\varepsilon = 10^{-6}$.

We performed numerical calculations, using the formula (13), of the distribution of the dimensionless temperature $T(\rho,\ \text{Fo}) = \frac{t(r,\tau)}{t_{\text{in}}}$ as a function of the coordinate $\rho = r/R$ and

Fourier's number Fo for different values of Biot's number Bi. In the calculations we took $t_{in} = 20°C$ and $t_0 = 300°C$. The results are presented in the form of plots in Figs. 1 and 2. The solid curves correspond to the solution of the nonlinear boundary-value problem of heat conduction with Kirchhoff's variable, calculated from the formula (9), while the dot-dashed curves were calculated with Kirchhoff's variable taken in the form (10), i.e., they represent the approximate solution. The dashed lines correspond to the solution of the linear boundary-value problem, i.e., under the assumption that the thermophysical characteristics do not depend on the temperature.

The numerical calculations showed that when the temperature dependence of the thermophysical characteristics is taken into account the temperature is lower than the corresponding temperatures in a uniform heat-insensitive body. It should also be noted that linearizing the boundary condition of the third kind, performed by simply replacing t with ϑ, results in a significant distortion of the behavior of the temperature in the heat-sesitive sphere even in the case when the characteristics are linear functions of the temperature with a comparatively small value of the coefficient k_0.

NOTATION

t, temperature field; ϑ, Kirchhoff's variable; λ, thermal conductivity; c_V, heat capacity at constant volume; a, heat-transfer coefficient of the surface $r = R$; $S_{\pm}(\xi)$, asymmetric unit step functions; τ, time; and r, radial coordinate.

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VARIATIONAL FORMULATION OF A NONSTATIONARY HEAT-CONDUCTION PROBLEM

V. A. Bondarev

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A functional for a heat conduction problem is determined from thermodynamic considerations; also a class of functions is determined in which it is possible to realize an extremum of the functional.

According to the second law of thermodynamics, in a nonstationary thermodynamical physical system only such spontaneous processes are possible which bring a system of bodies participating in heat exchange closer to equilibrium. In addition, the rate of approach to equilibrium is determined by an increase of entropy s, which reaches a maximum value in the state of equilibrium. In the case of a variational formulation of a nonstationary heat conduction problem, arbitrarily small variations in the solution ϑ can be regarded as the result of the action of some fictitious sources of heat in the system. If the solution ϑ is varied in such a way that the fictitious forces lead to a change of entropy in a direction away from equilibrium with respect to real processes, then, for the same initial and boundary conditions, the time for the system to approach equilibrium will increase. Then the largest rate of approach to equilibrium will correspond to the solution ϑ.

In view of the above, it is appropriate to assume that for the nonstationary heat conduction problem a functional can be constructed having an extremum at the solution ϑ, real-

Belorus Polytechnic Academy, Minsk.

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